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4 SEM TDC MTMH (CBCS) C 10

2022

(June/July)

MATHEMATICS

(Core)

Paper : C–10

(Ring Theory and Linear Algebra—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. (a) Give an example of a ring without unity. 1

(b) Define unit element in a ring. 1

(c) If the unity and the zero element of a ring R are equal, show that $R = \{0\}$, where 0 is the zero element of R. 2

(d) Give an example of a subring which is not an ideal. 2

(e) If I is an ideal of a ring R with unity such that $1 \in I$, show that $I = R$. 2

(f) Show that $\mathbb{Z}_{12}$ is not an integral domain. 2

(g) Show that every field is an integral domain. Give an example to show that every integral domain is not necessarily a field. 4+1=5

Or

Define characteristic of a ring. Prove that the characteristic of an integral domain is 0 or a prime. 1+4=5

(h) Show that if A and B are two ideals of a ring R, then $A+B$ is an ideal of R containing both A and B, where

$$A+B=\{a+b \mid a \in A, b \in B\}$$ 5

Or

Show that in a Boolean ring R, every prime ideal $P \neq R$ is maximal. 5

2. *(a)* Define kernel of a ring homomorphism. 1

(b) If $f: R \to R'$ be a ring homomorphism, show that $f(-a) = -f(a)$. 2

(c) Let R be a commutative ring with char $(R) = 2$. Show that $\phi : R \to R$ defined by $\phi(x) = x^2$ is a ring homomorphism. 2

(d) Let

$$R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{Z} \right\} \text{ and } \phi : \mathbb{R} \to \mathbb{Z}$$

defined by

$$\phi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) = a - b$$

Find $\ker \phi$. 2

(e) Let $f: R \to R'$ be an onto homomorphism, where R is a ring with unity. Show that $f(1)$ is the unity of R'. 3

Or

Prove that a homomorphism $f: R \to R'$ is one-one if and only if $\ker f = \{0\}$. 3

(f) Show that the relation of isomorphism in rings is an equivalence relation. 5

Or

Let A, B be two ideals of a ring R. Show that

$$\frac{A+B}{A} \cong \frac{B}{A \cap B}$$ 5

3. (a) Is $\mathbb{R}$ a vector space over $\mathbb{C}$? 1

(b) Define zero subspace of a vector space. 1

(c) For $x=(x_1, x_2)$ and $y=(y_1, y_2)$ of $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$, let $x+y=(x_1+y_1, x_2+y_2)$ and $\alpha x = \alpha(x_1, x_2) = (\alpha x_1, 0)$. Is $\mathbb{R}^2$ a vector space with respect to above operations? Justify your answer. 1+1=2

(d) Let V be a vector space of all 2×2 matrices over the field $\mathbb{R}$ of real numbers. Show that the set S of all 2×2 singular matrices over $\mathbb{R}$ is not a subspace of V. 2

(e) Consider the vectors $v_1 = (1, 2, 3)$ and $v_2 = (2, 3, 1)$ in $\mathbb{R}^3(\mathbb{R})$. Find k so that $u = (1, k, 4)$ is a linear combination of v_1 and v_2. 2

(f) Show that the vectors $v_1 = (1, 1, 0)$, $v_2 = (1, 3, 2)$ and $v_3 = (4, 9, 5)$ are linearly dependent in $\mathbb{R}^3(\mathbb{R})$. 3

(g) Prove that any basis of a finite-dimensional vector space is finite. 4

Or

Let W_1 and W_2 be two subspaces of a finite-dimensional vector space. Then show that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$ 4

4. *(a)* Let T be a linear transformation from a vector space U to a vector space V over the field F. Prove that the range of T is a subspace of V. 3

(b) Examine whether the following mappings are linear or not : 2+2=4

(i) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (|x|, y + z)$$

(ii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x, y) = (x + y, x)$$

(c) If $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(x, y) = (x + y, x - y, y)$$

is a linear transformation, find the rank and nullity of T. 4+4=8

(d) Let T be a linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, 0)$. Find the matrix of T with respect to the basis $\{v_1, v_2\}$, where $v_1 = (1, 1)$ and $v_2 = (2, -1)$. 5

(e) Let $T: V \to U$ be a linear transformation. Show that

$$\dim V = \text{rank } T + \text{nullity } T$$

5

Or

Prove that a linear transformation $T: V \to U$ is non-singular if and only if T carries each linearly independent subset of V onto a linear independent subset of U. 5

(f) Define isomorphism of vector spaces. Prove that the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \vdash (a, b, c, d)$$

from $M_2(\mathbb{R})$ to $\mathbb{R}^4$ is an isomorphism. 5

Or

Prove that every n-dimensional vector space over a field F is isomorphic to F^n. 5

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