6 SEM TDC MTMH (CBCS) C 13

2022

(June/July)

MATHEMATICS

(Core)

Paper : C–13

(Metric Spaces and Complex Analysis)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Every non-empty set can be regarded as a metric space. State true or false. 1

   (b) Write when a metric is called a discrete metric. 1

   (c) Write the definition of an open set in metric space. 2

   (d) Define complete metric space. 2

(e) If (X, d) is a metric space and $x, y, z \in X$ be any three distinct points, then show that $d(x, y) \geq |d(x, z) - d(z, y)|$. 4

(f) Answer any *two* from the following : 5×2=10

(i) Prove that in any metric space X, each open sphere is an open set.

(ii) Let X be any non-empty set and d a function defined on X, such that $d : X \times X \to R$ defined by

$$d(x, y) = 0 \text{ , if } x = y$$
$$= 1 \text{ , if } x \neq y$$

Prove that d is a metric on X.

(iii) If (X, d) be a metric space and $\{x_n\}$, $\{y_n\}$ are sequences in X such that $x_n \to x$ and $y_n \to y$, then show that

$$\{d(x_n, y_n)\} \to d(x, y)$$

(iv) Prove that the limit of a sequence in a metric space, if it exists, is unique.

2. *(a)* Real line R is not connected. State true or false. 1

(b) Write one property of continuous mapping. 1

(c) Write the definition of uniform continuity in a metric space. 2

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(f) Prove that $f(z) = z^2 + 2z + 3$ is continuous everywhere in the finite plane. 5

Or

Prove that if $w = f(z) = u + iv$ is analytic in a region R, then

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

4. (a) Define an analytic function at a point. 1

(b) Write the interval of θ in the principal value of $\log z = \log r + i\theta$. 1

(c) Write $\sinh z$ in terms of exponential functions. 1

(d) Write the value of $\int_C dz$ where C is a closed curve. 1

(e) Show that the function $f(z) = e^{x+iy}$ is analytic. 4

(f) Find

$$\int_0^1 ze^{2z}\,dz$$ 4

Or

Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by $z = t^2 + it$.

5. (a) Obtain Taylor's series for the function

$$f(z)=\frac{(z-2)(z+2)}{(z+1)(z+4)}$$

when $|z|<1$. 4

(b) State and prove Liouville's theorem. 6

Or

Prove that the series

$$z(1-z)+z^2(1-z)+z^3(1-z)+\cdots$$

converges for $|z|<1$.

6. (a) Write the statement of Laurent's theorem. 2

(b) Expand

$$f(z)=\frac{1}{(z+1)(z+3)}$$

in a Laurent series valid for $1<|z|<3$. 6

Or

Prove that the sequence $\left\{\frac{1}{1+nz}\right\}$ is uniformly convergent to zero for all z such that $|z|\geq 2$.

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