**3 SEM TDC MTMH (CBCS) C 5**

**2021**

( Held in January/February, 2022 )

# MATHEMATICS

( Core )

Paper : C–5

**( Theory of Real Functions )**

*Full Marks* : 80
*Pass Marks* : 32

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

1. (a) Define limit of function at a point. 1

   (b) Evaluate the following limits (any *one*) : 2

   (i) $\lim_{x \to 2} \sqrt{\dfrac{2x+1}{x+3}}$

   (ii) $\lim_{x \to 1} \dfrac{x-1}{\sqrt{x+3}-2}$

(c) If $f : A \to R$ and if $c$ is a cluster point of $A$, then prove that $f$ can have only one limit at $c$. 3

2. (a) Write the type of discontinuity if

$$\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$$ 1

(b) When does a function $f$ continuous on a set? 2

(c) Investigate for the point of discontinuity : 4

$$f(x) = \begin{cases} 1 \; ; & \text{if } x \text{ is rational} \\ 0 \; ; & \text{if } x \text{ is irrational} \end{cases}$$

*Or*

Let $A, B \subseteq R$ and let $f : A \to R$ and $g : B \to R$ be functions such that $f(A) \subseteq B$. If $f$ is continuous at a point $c \in A$ and $g$ is continuous at $b = f(c) \in B$; then prove that composition $g \circ f : A \to R$ is continuous at $c$.

(d) Let $A \subseteq R$, let $f : A \to R$ and let $|f|$ be defined by $|f|(x) = |f(x)|$ for $x \in A$ and $f$ is continuous at a point $c \in A$. Prove that $|f|$ is continuous at $c$. 3

*Or*

Discuss the continuity of $f(x) = |x-1| + |x-2|$ in the interval [0, 3].

3\. (a) State location of roots theorem. 1

(b) State and prove intermediate value theorem. 4

(c) Find the roots of the equation $x^3 - x - 1 = 0$ between 1 and 2 by using location of roots (bisection method) theorem. 4

*Or*

Let $I$ be a closed bounded interval and let $f : I \to R$ be continuous on $I$, then prove that the set $f(I) = \{f(x) : x \in I\}$ is a closed bounded interval.

4\. (a) Write the non-uniformity continuity criteria (any one). 1

(b) Show that a function $f : R \to R$ given by $f(x) = x^2$ is not uniformly continuous on $R$. 4

*Or*

If $f$ and $g$ are each uniformly continuous on $R$, then prove that composite function $f \circ g$ is also uniformly continuous on $R$.

5. (a) Find :

$$\frac{d}{dx}(\tan x^2)$$ 1

(b) State Caratheodory's theorem. 2

(c) If $f$ is continuous on the closed interval $I = [a, b]$ and $f$ is differentiable on the open interval $(a, b)$ and $f'(x) = 0$ for all $x \in (a, b)$, prove that $f$ is constant on $I$. 3

6. (a) Define relative maximum and relative minimum at a point on an interval. 2

(b) State and prove Rolle's theorem. 1+3=4 *(c)* Apply the mean value theorem to prove the following (any *one*) : 4

(*i*) $e^x \geq 1 + x$ for $x \in R$

(*ii*) $\dfrac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b-a}{1+a^2}$

for $a < b$

**7.** *(a)* Show that $f(x) = x^3 - 3x^2 + 3x + 2$ is strictly increasing for every value of $x \in R$ except 1. 2

*(b)* Let $I \subseteq R$ be an interval, let $f : I \to R$, let $c \in I$ and assume that $f$ has a derivative at $c$ and $f'(c) > 0$, then there is a number $\delta > 0$. Prove that $f(x) > f(c)$ for $x \in I$ and $c < x < c + \delta$. 3

*(c)* Examine the validity of mean value theorem for the function $f(x) = 2x^2 - 7x + 10$ on $[2, 5]$. 4

*Or*

If $f$ is differentiable on $I = [a, b]$ and if $k$ is a number between $f'(a)$ and $f'(b)$, then prove that there exists at least one point $c$ in $(a, b)$, where $f'(c) = k$.

8. (a) Write the remainder after $n$ terms of Taylor's theorem in Lagrange's form. 1

(b) Write the statement of Cauchy's mean value theorem. 2

(c) Deduce from Cauchy's mean value theorem $f(b) - f(a) = \xi f'(\xi) \log \frac{b}{a}$, where $f(x)$ is continuous and differentiable in $[a, b]$ and $a < \xi < b$. 3

(d) State and prove Taylor's theorem with Cauchy's form of remainder. 6

*Or*

Find the approximate value of $\sqrt[3]{1+x}$, $x > -1$ by using Taylor's theorem with $n = 2$.

9. (a) Write the necessary condition for a function $f(x)$ to have relative extremum at $x = c$. 1

(b) Determine whether or not $x = 0$ is a point of relative extremum of $f(x) = \sin x - x$. 2

(c) Define convex function. 2

(d) Using Maclaurin's series, expand the following in an infinite series in powers of $x$ (any *two*) : 4×2=8

(i) $\log(1+x)$

(ii) $\cos x$

(iii) $\dfrac{1}{ax+b}$

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